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# AN ESTIMATE FOR HEXAGONAL CIRCLE PACKINGS

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## 1. Introduction

Let $P$ be a circle packing in the complex plane $\mathbf{C}$, i.e., a collection of circles in $\mathbf{C}$ with disjoint interiors, and let $c_0$ be a circle of $P$. Suppose that for some positive integer $n \geq 2$, the $n$ generations $P_n$ of $P$ about $c_0$ (defined successively by $P_0 = \{c_0\}$, $P_k = \{c \in P; c \in P_{k-1}$ or $c$ is tangent to some circle of $P_{k-1}\}$, $k \geq 1$) is combinatorially equivalent to the $n$ generations $H_n$ of a regular hexagonal circle packing about one of its circles. Then the ratio of radii of any two circles of $P$ tangent to $c_0$ is bounded by $1 + s_n$, where $s_2, s_3, \ldots$ is some decreasing sequence of positive numbers. We will denote by $s_n$ the smallest possible constant with this property. In [7], B. Rodin and D. Sullivan showed that any circle packing which is combinatorially equivalent to an infinite regular hexagonal circle packing is also regular hexagonal, and as a consequence, $s_n$ converges to $0$. They conjectured that $s_n \leq C/n$ for some constant $C$. In this paper, we will prove this conjecture. This estimate for $s_n$ is best possible as (we will see later) $s_n \geq 4/n$.

One may use our result to estimate the rate of convergence of the circle packing solutions $f_\varepsilon$ to the Riemann Mapping Theorem given in [7], where $\varepsilon$ is the size of the preimage circles, and of the approximating solutions $f_\delta$ to the Beltrami equations constructed in [4]. This shows that these solutions are constructive. Moreover, for the circle packing solutions $f_\varepsilon$ of [7], we may combine with [6, Theorems 5 and 8] to conclude that the rate of convergence on compact subsets is of order at most $\varepsilon^{\alpha/8}$ for any fixed $\alpha < 1$, and their derivatives converge in $L^\infty$ on compact subsets.

The proof of $s_n \leq C/n$ will be given in §2 with the assistance of an area estimate on the union of the images of the interstices bounded by the circles of $H_n$ under the Schottky group generated by inversions of the circles of $H_n$ (Lemma 2.2). In §3 we will prove this estimate. The argument also leads to vanishing of the Lebesgue measure of the limit

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set of a class of infinitely generated Schottky groups. §4 is independent of the main subject, and discusses the globally uniform convergence of quasiconformal mappings. We will see that when the domain $R$ (or $\Omega$ in [4] ) is a Jordan domain, $f_\varepsilon$ of [7] (or $f_\delta$ of [4]) converges globally uniformly.

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## 2. Proof of the estimate $s_n \leq C/n$

For any positive integer $n$, let $H_n$ be $n$ generations of some regular hexagonal circle packing about one of its circles, say $c_0$. We may normalize $H_n$ so that $c_0$ is the unit circle and 1 is a point of tangency of $c_0$ with some neighboring circle. In this way, the circles of $H_n$ have radius 1 and are centered at points $2(k_1 + k_3) + 2e^{\pi i/3}(k_2 - k_3)$, where $k_1$, $k_2$, and $k_3$ are integers with $|k_1| + |k_2| + |k_3| \leq n$.

Let $H'_n$ be $n$ generations of a circle pacing $P$ about some circle $c'_0$ of $P$ such that $H'_n$ is combinatorially equivalent to $H_n$. By this, we mean that there is a one-to-one correspondence of $H'_n$ and $H_n$ so that two circles of $H'_n$ are tangent if and only if their corresponding circles in $H_n$ are tangent. Let $c'_1, c'_2, \cdots, c'_6$ be the six circles of $H'_n$ corresponding to the circles $c_k = \{|z - 2e^{\pi(k-1)i/3}| = 1\}$, $k = 1, 2, \cdots, 6$, of $H_n$ which are tangent to $c_0 = \{|z| = 1\}$. Then

$$s_n = \sup_{(P, c_0)} \max_{1 \leq j, k \leq 6} \left( \frac{\text{radius}(c'_j)}{\text{radius}(c'_k)} - 1 \right), \tag{2.1}$$

where $(P, c_0)$ is any pair satisfying the above property.

The estimation of $s_n$ is briefly described as follows. First, there is a quasiconformal mapping $\psi$ from plane to plane which maps the subpacking $H_m$ ($m \sim n/2$ for $n$ large) of $H_n$ to the corresponding subpacking $H'_m$ of $H'_n$. This mapping will be made conformal on the union $I_m$ of interstices bounded by the circles of $H_m$. Normalize $H'_n$ so that $c'_0 = c_0$

and $c_0' \cap c_1' = c_0 \cap c_1 = \{1\}$. We wish to show that $\psi$ restricted to $c_0$ is $O(1/m)$–close to the identity (and this implies that the points of tangency $c_0' \cap c_j'$, $j = 1, 2, \cdots, 6$, are almost equidistributed on the unit circle $c_0'$, a fact which clearly yields an estimate for $s_n$). Using the Schottky group $G_m$ generated by inversions on the circles of $H_m$, one may modify $\psi$ in the interiors of circles of $H_m$ so it becomes conformal on the images of $I_m$ by the transformations of $G_m$. These images fill up most of the area of the unit disk $D$ ($=$ interior of $c_0$). As a result, the above quasiconformal mapping restricted to $D$ (which maps $D$ onto $D =$ interior of $c_0'$) will be close to the identity. It follows that $\psi|c_0'$ is close to the identity.

In the following, let us denote by $\delta_j$ and $C_j$, $j = 1, 2, \ldots$, some positive universal constants. $\delta_j$ will be used for lower bounds and $C_j$ for upper bounds, so we will always assume $0 < \delta_j \leq 1$ and $1 \leq C_j < \infty$.

Let $\delta_1$ be the constant in the Ring Lemma of [7, §4] for hexagonal packing. This means that if six circles surround a circle of radius $r$, then each circle has radius at least $\delta_1 r$.

**Lemma 2.1.** *For any three mutually tangent circles $c_0'$, $c_1'$, and $c_2'$ with disjoint interiors such that the ratio of the radii of any two circles is between $\delta_1$ and $1/\delta_1$, there is an orientation-preserving Möbius transformation $g$ which maps $c_0 = \{|z| = 1\}$, $c_1 = \{|z-2| = 1\}$, and $c_2 = \{|z - 2e^{\pi i/3}| = 1\}$ onto $c_0'$, $c_1'$, and $c_2'$ respectively. Moreover, $g$ is $C_1$-bi-Lipschitz on $c_0$ if $c_0'$ if normalized to have radius $1$.*

*Proof.* Let $g$ be the orientation-preserving Möbius transformation sending $c_j \cap c_k$ to $c_j' \cap c_k'$, where $(j, k)$ is any pair of $\{(0, 1), (0, 2), (1, 2)\}$. Then $g$ satisfies the requirements of the lemma. q.e.d.

Note that $g$ maps the interstice bounded by $c_0$, $c_1$, and $c_2$ to that bounded by $c_0'$, $c_1'$, and $c_2'$. Now for any three mutually tangent circles in $H_{n-1}$, the Ring Lemma of [7] implies that the corresponding circles in $H_{n-1}'$ satisfy the conditions of Lemma 2.1, so there is a conformal mapping from each interstice bounded by circles of $H_{n-1}$ to the interstice bounded by corresponding circles of $H_{n-1}'$. These conformal mappings may be glued together to form a conformal mapping from the union of interstices bounded by circles of $H_{n-1}$ to the union of interstices bounded by circles of $H_{n-1}'$. Furthermore, this mapping maps each circle of $H_{n-2}$ to the corresponding circle of $H_{n-2}'$ and, by Lemma 2.1, it is $C_1$-bi-Lipschitz if both circles were normalized. So we can extend the mapping radially on each disk bounded by circles of $H_{n-2}$ and the result is a $C_1$-quasiconformal mapping $\varphi$ from the union of interstices and disks bounded by circles of $H_{n-2}$ to the corresponding union bounded by circles of $H_{n-2}'$.

It is well known (see e.g. [2, p. 96]) that for any $C_1$-quasiconformal mapping of the unit disk to some region in $\mathbf{C}$, its restriction to $\{|z| \le 1/\sqrt{3}\}$ may be extended to some $C_2$-quasiconformal homeomorphism of $\mathbf{C}$. As the union of interstices and disks bounded by circles of $H_{n-2}$ contains the disk $\{|z| < (n-2)\sqrt{3}\}$, the restriction of $\varphi$ to $\{|z| \le n-2\}$ has a $C_2$-quasiconformal extension $\psi: \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ with

$$\psi(\infty) = \infty. \tag{2.2}$$

Let

$$m = m(n) = \left[\frac{n-3}{2}\right] = \text{ the integer part of } \frac{n-3}{2}. \tag{2.3}$$

Then all circles of $H_m$ lie in $\{|z| \le n-2\}$; it follows that $\psi$ equals $\varphi$ on the union of interstices and disks bounded by circles of $H_m$. Particularly, $\psi$ is conformal on the union $I_m$ of interstices bounded by the circles of $H_m$.

For any circle $c$ in $\widehat{\mathbf{C}}$, we will denote by $\gamma_c$ the inversion on $c$. Let $c$ be a circle of $H_m$, let $\Delta$ be the disk bounded by $c$, and let $c'$ be the circle of $H'_m$ which corresponds to $c$. Then since $\psi$ maps $c$ onto $c'$, we may replace $\psi|\Delta$ by $\gamma_{c'} \circ \psi \circ \gamma_c|\Delta$. When this is done for each $c \in H_m$, we obtain a $C_2$-quasiconformal mapping $\psi^1: \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ which is conformal on

$$I_m^1 = I_m \cup \left( \bigcup_{c \in H_m} \gamma_c(I_m) \right), \tag{2.4}$$

and maps each circle of

$$H_m^1 = \bigcup_{c \in H_m} \gamma_c(H_m \backslash \{c\}) \tag{2.5}$$

to a corresponding circle of

$$H_m^{'1} = \bigcup_{c' \in H'_m} \gamma_{c'} \backslash \{c'\}).$$

Similarly, for each circle $c$ of $H_m^1$, let $\Delta$ be the disk it bounds, let $c'$ be the corresponding circle of $H_m^{'1}$, and replace $\psi^1|\Delta$ by $\gamma_{c'} \circ \psi^1 \circ \gamma_c|\Delta$. We obtain a $C_2$-quasiconformal mapping $\psi^2: \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ which is conformal on $I_m^2 = I_m^1 \cup (\bigcup_{c \in H_m^1} \gamma_c(I_m^1))$, and maps each circle of $H_m^2 = \bigcup_{c \in H_m^1} \gamma_c(H_m^1 \backslash \{c\})$ to the corresponding circle of $H_m^{'2} = \bigcup_{c' \in H_m^{'1}} \gamma_{c'}(H_m^{'1} \backslash \{c'\})$. Continuing in this way, we may find for each $k$ a $C_2$-quasiconformal mapping $\psi^k : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ which is conformal on

$$I_m^k = I_m^{k-1} \cup \left( \bigcup_{c \in H_m^{k-1}} \gamma_c(I_m^{k-1}) \right), \tag{2.6}$$

and maps each circle of

$$H_m^k = \bigcup_{c \in H_m^{k-1}} \gamma_c(H_m^{k-1} \backslash \{c\}) \tag{2.7}$$

to the corresponding circle of

$$H_m^{'k} = \bigcup_{c' \in H_m^{'k-1}} \gamma_{c'}(H_m^{'k-1} \backslash \{c'\}).$$

It is easy to see that $\psi^k$ converges to some $C_2$-quasiconformal mapping $f : \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ which is conformal on the set

$$J_m = \bigcup_{k=1}^{\infty} I_m^k. \tag{2.8}$$

Note that the set $J_m$ is equal to the union of images of $I_m$ by the elements of the Schottky group $G_m$ generated by the inversions $\gamma_c$, $c \in H_m$. This implies particularly that $\gamma(J_m) = J_m$ for $\gamma \in G_m$.

From (2.2), we see that $\psi^1(0) = \gamma_{c_0'} \circ \psi \circ \gamma_{c_0}(0) = \gamma_{c_0'}(\infty)$ is the center of $c_0'$. We may normalize $H_n'$ so that $c_0' = c_0 =$ the unit circle, and $c_0'$ is tangent to $c_1'$ at $1$. Then $\psi^1(0) = 0$ and $\psi(1) = 1$, from which it follows that $\psi^k(0) = 0$ $\forall k \geq 2$ and hence that

$$f(0) = 0. \tag{2.9}$$

On the other hand, $f(e^{i\theta}) = \psi^k(e^{i\theta}) = \psi(e^{i\theta})$ for any $\theta \in \mathbf{R}$, particularly,

$$f(1) = \psi(1) = 1. \tag{2.10}$$

The following lemma gives a critical estimate on the measure of the set $D \backslash J_m$. Its proof will be given in §3.

**Lemma 2.2.** *For each positive integer $m$ we have*

$$|D \backslash J_m| \leq C_3/m^2, \tag{2.11}$$

*where $|\cdot|$ denotes the Lebesgue measure in the plane.*

Consider the restriction on the unit disk $D$ of $f$, still denoted by $f$. Then $f$ is a $C_2$-quasiconformal self-homeomorphism of $D$. We wish to show that $f$ (and hence $\psi$) restricted to the unit circle $\partial D = c_0$ is $O(1/n)$-close to the identity id. The following is an argument based on an idea of Dennis Sullivan. Consider the Riemann sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ endowed with the spherical metric induced by stereographic projection. Define $F: \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ by "doubling" $f$:

$$F(z) = \begin{cases} f(z) & \text{if } |z| \le 1, \\ 1/\overline{f(1/\overline{z})} & \text{if } |z| \ge 1. \end{cases} \tag{2.12}$$

Then $F$ is a $C_2$-quasiconformal self-homeomorphism of $\widehat{\mathbf{C}}$. By (2.9) and (2.10), $F$ fixes $0$, $1$, and $\infty$. On the other hand, Lemma 2.2 implies that $F$ is conformal except on a subset of spherical area $\le O(1/m^2)$.

Take a point $z_0$ in $\widehat{\mathbf{C}}$ whose spherical distance from $0$, $1$, and $\infty$ is uniformly bounded from below, say $\ge 1/10$. Then $F$ maps the four-punctured sphere $\widehat{\mathbf{C}}\backslash\{0, 1, \infty, z_0\}$ onto the four-punctured sphere $\widehat{\mathbf{C}}\backslash\{0, 1, \infty, F(z_0)\}$. These punctured spheres are doubly covered (via some elliptic functions $\pi_1$ and $\pi_2$) by some four-punctured tori $T_1$ and $T_2$, respectively. Then $F$ lifts to a $C_2$-quasiconformal homeomorphism $\overline{F}$ of $T_1$ and $T_2$ which is conformal except on the preimage by $\pi_1$ of the set of nonconformality of $F$. This last set also has spherical area $\le O(1/m^2)$, and the covering mapping $\pi_1$ behaves like $z \to z^2$ near each of the four punctures. So, if $T_1$ is endowed with the flat metric of total volume uniformly bounded from above, the area of the subset of $T_1$ where $\overline{F}$ fails to be conformal is bounded by $(O(1/m^2))^{1/2} = O(1/m)$.

Let $\zeta_1$ and $\zeta_2$ be the conformal moduli of $T_1$ and $T_2$ respectively. Then we may identify $T_j$ with $\mathbf{C}/(z \sim z+1 \sim z+\zeta_j)$, $j = 1, 2$. Since $z_0$ is bounded away from $0$, $1$, and $\infty$, $\zeta_1$ should fall into a compact subset of the upper half-plane. We claim that

$$|\zeta_2 - \zeta_1| \le O(1/m). \tag{2.13}$$

In fact, let $\widetilde{F}: \mathbf{C} \to \mathbf{C}$ be the lift of $\overline{F}: T_1 \to T_2$. Then $\widetilde{F}(z+1) = \widetilde{F}(z)+1$ and $\widetilde{F}(z+\zeta_1) = \widetilde{F}(z) + \zeta_2$. Let $K: \mathbf{C} \to [1, C_2]$ be the pointwise linear dilatation of $\widetilde{F}$. Then we have

$$\begin{aligned} 1 = \widetilde{F}(iy+1) - \widetilde{F}(iy) &= \int_0^1 \frac{\partial \widetilde{F}}{\partial x}(x+iy)\,dx \\ &\le \int_0^1 \left|\frac{\partial \widetilde{F}}{\partial x}(x+iy)\right| dx \le \int_0^1 K(x+iy)^{1/2} J(x+iy)^{1/2}\,dx, \end{aligned}$$

where $J(x+iy)$ denotes the Jacobian of $\widetilde{F}$ as a function from $\mathbf{R}^2$ to $\mathbf{R}^2$. Integrate the above inequality over $y \in [0, y_1]$, where $y_1 = \mathrm{Im}(\zeta_1)$. We get

$$y_1 \le \int_0^{y_1}\int_0^1 K^{1/2}J^{1/2}\,dx\,dy = \iint_{T_1} K^{1/2}J^{1/2}\,dA.$$

Using the Schwarz inequality, we obtain

$$y_1^2 \le \iint_{T_1} K\,dA \iint_{T_1} J\,dA = \iint_{T_1} K\,dA\cdot \mathrm{Area}(T_2) = \iint_{T_1} K\,dA\cdot(\mathrm{Im}(\zeta_1)).$$

Recall that $K=1$ on $T_1$ except on a subset of area $\le O(1/m)$. Therefore

$$\iint_{T_1} K\,dA = \mathrm{Area}\,T_1 + \iint (K-1)\,dA \le \mathrm{Im}(\zeta_1) + (C_2-1)O(1/m),$$

which implies

$$\mathrm{Im}(\zeta_1)^2 = y_1^2 \le [\mathrm{Im}(\zeta_1)+O(1/m)]\cdot \mathrm{Im}(\zeta_2).$$

As $\mathrm{Im}(\zeta_1)\in(0,\infty)$ lies on a compact subset, we obtain

$$\mathrm{Im}(\zeta_1) \le \mathrm{Im}\,\zeta_2 + O(1/m). \tag{2.14}$$

Similarly, let $\alpha_1$ and $\alpha_2$ be integers. Then

$$\begin{aligned}|\alpha_1+\alpha_2\zeta_1| &\le \int_0^1 \left|\frac{\partial \widetilde{F}(x+t(\alpha_1+\alpha_2\zeta_1))}{\partial t}\right| dt\\ &\le |\alpha_1+\alpha_2\zeta_1| \int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\\ &\quad\cdot J(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\,dt.\end{aligned}$$

Integrating this inequality over $x\,\int[0,1]$ yields

$$\begin{aligned}|\alpha_1+\alpha_2\zeta_2| \le |\alpha_1+\alpha_2\zeta_1| \int_0^1\int_0^1 & K(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\\ &\cdot J(x+t(\alpha_1+\alpha_2\zeta_1))^{1/2}\,dt\,dx.\end{aligned}$$

Then by the Schwarz inequality, we find

$$\begin{aligned}|\alpha_1+\alpha_2\zeta_2|^2 &\le |\alpha_1+\alpha_2\zeta_1|^2 \int_0^1\int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))\,dt\,dx\\ &\quad\cdot \int_0^1\int_0^1 J(x+t(\alpha_1+\alpha_2\zeta_1))\,dt\,dx\\ &= |\alpha_1+\alpha_2\zeta_1|^2 \frac{\mathrm{Im}(\zeta_2)}{\mathrm{Im}(\zeta_1)} \int_0^1\int_0^1 K(x+t(\alpha_1+\alpha_2\zeta_1))\,dt\,dx.\end{aligned}$$

Therefore

$$|\alpha_1 + \alpha_2\zeta_2|^2 \le |\alpha_1 + \alpha_2\zeta_1|^2 \frac{\mathrm{Im}(\zeta_2)}{\mathrm{Im}(\zeta_1)}(1 + O(1/m)),$$

which implies

$$\frac{|\alpha_1 + \alpha_2\zeta_2|^2}{\mathrm{Im}(\zeta_2)} \le \frac{|\alpha_1 + \alpha_2\zeta_1|^2}{\mathrm{Im}(\zeta_1)}(1 + O(1/m)). \tag{2.15}$$

Note that (2.15) holds for any rationals $\alpha_1$ and $\alpha_2$ and hence for any $\alpha_1$, $\alpha_2 \in \mathbf{R}$. Take $\alpha_1 = -\mathrm{Re}(\zeta_1)$, $\alpha_2 = 1$, and obtain

$$\mathrm{Im}(\zeta_2) \le \mathrm{Im}(\zeta_1)(1 + O(1/m)),$$

which together with (2.14) yields

$$|\mathrm{Im}(\zeta_2) - \mathrm{Im}(\zeta_1)| \le O(1/m). \tag{2.16}$$

Thus using (2.16), from (2.15) we deduce that

$$|\alpha_1 + \alpha_2\zeta_2|^2 \le |\alpha_1 + \alpha_2\zeta_1|^2(1 + O(1/m)),$$

which together with (2.16) implies $|\zeta_2 - \zeta_1| \le O(1/m)$.

Since $z_0$ and $F(z_0)$ depend smoothly (in fact analytically) on $\zeta_1$ and $\zeta_2$ respectively, from the closeness of $\zeta_2$ and $\zeta_1$ we obtain $|F(z_0) - z_0| \le O(1/m)$. Taking $z_0 = -1$, we get

$$|F(-1) - (-1)| \le O(1/m). \tag{2.17}$$

Now let $z_0$ be an arbitrary point on $\widehat{\mathbf{C}}$. If $z_0$ is bounded away from $0, 1, \infty$, we have shown that $|F(z_0) - z_0| \le O(1/m)$. But if $z_0$ is close to one of the points $0$, $1$, or $\infty$, say $0$, then we may apply the above argument to the four-punctured spheres $\widehat{\mathbf{C}}\backslash\{-1, 1, \infty, z_0\}$ and $\widehat{\mathbf{C}}\backslash\{F(-1), 1, \infty, F(z_0)\}$ to conclude that the cross ratio of $(F(-1), 1, \infty, F(z_0))$ is $O(1/m)$-close to the cross ratio of $(-1, 1, \infty, z_0)$, a fact which implies $|F(z_0) - z_0| \le O(1/m)$ by (2.17). In this way, we conclude that $\widetilde{F}$ is $O(1/m)$-close to the identity, and $f$ is $O(1/m)$-close to the identity. Therefore, $\psi$ is $O(1/m)$-close to the identity on $c_0$ and $s_n \le O(1/m)$. Since $m \sim n/2$, the estimate follows.

**Remark 1.** There is an obvious lower bound for $s_n$. First note that $2n + 2$ lies in the unbounded component in the complement of the union of circles of $H_n$. The Möbius transformation

$$g: z \to \frac{2(n+1)z - 1}{2(n+1) - z}$$

sends $2n+2$ to $\infty$, and hence $g(H_n)$ is a circle packing in $\mathbf{C}$ combinatorially equivalent to $H_n$. By (2.1),

$$s_n \geq \frac{\text{radius}(g(c_1))}{\text{radius}(g(c_4))} - 1,$$

where we recall that $c_1 = \{|z-2| = 1\}$ and $c_4 = \{|z+2| = 1\}$. A direct computation yields

$$s_n \geq \frac{16n+16}{4n^2-1} > 4/n.$$

This shows that the $O(1/n)$-estimate for $s_n$ is best possible.

By our estimate on $s_n$, we obtain (see [6]):

**Corollary 2.3.** *The circle packing solutions $f_\varepsilon$ to the Riemann mapping given in* [7] *have derivatives which converge uniformly on compact subsets to the derivatives of the Riemann mapping.*

**Remark 2.** From the structure of the subset of $D$ where $f\colon D \to D$ fails to be conformal, one can prove that there is a Möbius transformation $h$ of the unit disk such that $h \circ f$ is $O(1/n^2)$-close to the identity, which is stronger than the $O(1/n)$-closeness. This fact leads us to conjecture the circle packing solution $f_\varepsilon$ of [7] has "second derivatives" defined in an appropriate sense, and they converge to the second derivatives of the Riemann mapping. This would lead to the discovery of the (existence and the) value of $\lim_{n\to\infty} ns_n$.

## 3. Estimation of $|D \backslash J_m|$

In this section we prove

**Lemma 3.1.** *For any* (*not necessarily orientation preserving*) *Möbius transformation $h$ of the unit disk $D$, we have*

$$|D\backslash h(J_m)| \leq C_3/m^2. \tag{3.1}$$

Taking $h = \text{id}$ in (3.1), we obtain Lemma 2.2.

We begin with

**Lemma 3.2.** *There is some $\delta_2 > 0$ such that for any Möbius transformation $h$ of $D$ we have*

$$|h(J_1)| \geq \delta_2 \pi. \tag{3.2}$$

*Proof.* First, observe that the disk $\Delta_{j\,j+1}$ bounded by the circle $c_{j\,j+1}$ passing through $c_0 \cap c_j$, $c_0 \cap c_{j+1}$, and $c_j \cap c_{j+1}$ is in $J_1$ (see Figure 1, next page). This follows from the fact that on the disk $\Delta_{j\,j+1}$, the Fuschian group generated by the inversions $\gamma_{c_0}$, $\gamma_{c_j}$, $\gamma_{c_{j+1}}$ has the interstice bounded

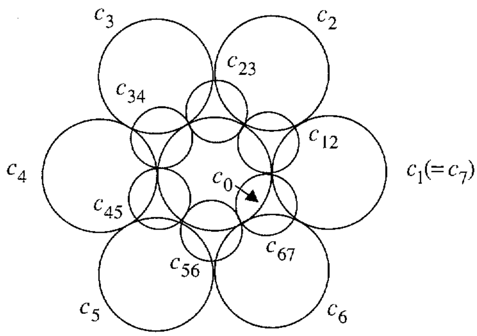


FIGURE 1

by the circles $c_0$, $c_j$, and $c_{j+1}$ as a fundamental domain. Thus, after applying a Möbius transformation $h$ of $D$, at least one image of the arcs $c_0 \cap \Delta_{j\,j+1}$ on $\partial D$ has length at least $\pi/3$. Then the image by $h$ of at least one of the regions $D \cap \Delta_{j\,j+1} \subseteq D \cap J_1$ has area bounded from below since $c_{j\,j+1}$ is orthogonal to $c_0$. This implies (3.2). q.e.d.

Consider the Schottky group $G_1$ generated by the inversions $\gamma_c$, $c \in H_1$. Let $U_1$ be the complement in $\widehat{\mathbf{C}}$ of the union of $I_1$ and the disks bounded by the circles of $H_1$. Then $I_1 \cup U_1$ is a finite-sided fundamental domain for $G_1$ and, by [2], the limit set of $G_1$ has measure zero. It follows particularly that for any Möbius transformation $h$ of $D$, we have

$$\left|\bigcup_{g\in G_1} (D \cap h \circ g(I_1))\right| + \sum_{h\in G_1} |D \cap h \circ g(U_1)| = \pi.$$

But $\bigcup_{g\in G_1} g(I_1) = J_1$, hence

$$|D \cap h(J_1)| + \sum_{g\in G_1} |D \cap h \circ g(U_1)| = \pi.$$

Using Lemma 3.2, this implies that

$$\sum_{g\in G_1} |D \cap h \circ g(U_1)| \le (1-\delta_2)\pi. \tag{3.3}$$

Lemma 3.1 will be proved by induction on $m$, with $C_3$ to be determined at the end of the proof. Obviously, (3.1) holds for $m = 1, 2$ if we choose $C_3 \ge 4\pi$. Now assume that (3.1) holds for $m \le l-1$. As $J_l$ is invariant by the maps of $G_l \supseteq G_1$, $D\backslash h(J_l) = h(D\backslash J_l)$ is equal (up to a measure zero subset) to the disjoint union of $D \cap h(g(U_1)\backslash J_l) = h(D \cap g(U_1\backslash J_l))$, $g \in G_1$. But for $g \in G_1$ we have either $g(U_1) \subseteq D$ or

$g(U_1) \cap D = \varnothing$, so

$$|D \backslash h(J_l)| = \sum_{\substack{g \in G_1 \\ g(U_1) \subseteq D}} |h \circ g(U_1 \backslash J_l)|. \tag{3.4}$$

On the other hand, (3.3) means that

$$\sum_{\substack{g \in G_1 \\ g(U_1) \subseteq D}} |h \circ g(U_1)| \le (1 - \delta_2)\pi. \tag{3.5}$$

So (3.1) follows if we prove

$$\frac{|h \circ g(U_1 \backslash J_l)|}{|h \circ g(U_1)|} \le \frac{C_3}{\pi(1 - \delta_2)/l^2} \tag{3.6}$$

for suitably chosen $C_3$. This is the goal of the next lemma.

**Lemma 3.3.** *Let $l \ge 3$. Suppose there is some $C_3 \ge 1$ so that* (3.1) *holds for any $m \le l - 2$. Then for any Möbius transformation $\gamma$ which maps $U_1$ to a bounded subset of* $\mathbf{C}$, *we have*

$$\frac{|\gamma(U_1 \backslash J_l)|}{|\gamma(U_1)|} \le \frac{9}{(1 - \sqrt{1 - \delta_2/2})^2 l^2} + \frac{C_3}{\pi(1 - \delta_2/2) l^2}. \tag{3.7}$$

*Proof.* $U_1$ is the union of $I_l \backslash I_1$, the disks $\Delta$ bounded by circles of $H_l \backslash H_1$ and the unbounded connected component $U_l$ in the complement of the circles of $H_l$. Define the "density" function $\eta_\gamma : \gamma(U_1) \to [0, 1]$ by

$$\eta_\gamma(z) = \begin{cases} 0 & \text{if } z \in \gamma(I_l \backslash I_1), \\ |\gamma(\Delta \backslash J_l)/|\gamma(\Delta)| & \text{if } z \in \gamma(\Delta),\ \partial\Delta \in H_l \backslash H_1, \\ 1 & \text{if } z \in \gamma(U_l). \end{cases}$$

Then

$$\frac{|\gamma(U_1 \backslash J_l)|}{|\gamma(U_1)|} = \frac{1}{|\gamma(U_1)|} \iint_{\gamma(U_1)} \eta_\gamma(z)\, dx\, dy. \tag{3.8}$$

Let $\Delta$ be a disk bounded by some circle $c = \partial\Delta$ of $H_k - H_{k-1}$, $2 \le k \le l - 1$, and let $z_\Delta$ be its center. Since the $l - k$ generations of the circle packing $H_l$ about $\partial\Delta$ is the translation $z_\Delta + H_{l-k}$, $z_\Delta + J_{l-k} \subseteq J_l$. Then by (3.1) for $m = l - k$ $(\le l - 2)$, we have

$$\gamma(\Delta \backslash J_l)| \le \frac{C_3}{\pi(l - k)^2} |\gamma(\Delta)|.$$

So if we define $\eta\colon U_1 \to [0, 1]$ by

$$\eta(z) = \begin{cases} 0 & \text{if } z \in I_l - I_1, \\ \min(C_3/\pi(l-k)^2, 1) & \text{if } z \in \Delta,\ \partial\Delta \in H_k - H_{k-1}, \\ & \quad 2 \le k \le l-1, \\ 1 & \text{if } z \in U_l \cup (\bigcup \Delta),\ \partial\Delta \in H_l - H_{l-1}, \end{cases}$$

then $\eta_\gamma(z) \le \eta(\gamma^{-1}(z))\ \ \forall z \in \gamma(U_1)$.

If $z \in \Delta$, $\partial\Delta \in H_k \backslash H_{k-1}$, $2 \le k \le l-1$, then $|z| \ge k\sqrt{3} - 1 > k$, and thus

$$\frac{C_3}{\pi(l-k)^2} \le \frac{C_3}{\pi[(l-|z|)^+]^2},$$

where $j^+ = \max(j, 0)$. Let $\rho\colon U \to [0, 1]$ be the following function:

$$\rho(z) = \rho(|z|) = \min\left(\frac{C_3}{\pi[(l-|z|)^+]^2}, 1\right). \tag{3.9}$$

Then $\eta \le \rho$ on $U_1$, and hence

$$\eta_\gamma \le \eta \circ \gamma^{-1} \le \rho \circ \gamma^{-1}(z). \tag{3.10}$$

Let $V = \{|z| > 3\} \cup \{\infty\}$. Then $V \subseteq U_1$, and for any $z \in U_1 \backslash V$, $z' \in V$, we have

$$\rho(z) \le \min\left(\frac{C_3}{\pi(l-3)^2}, 1\right) \le \rho(z').$$

Therefore

$$\frac{1}{|\gamma(U_1)|}\iint_{\gamma(U_1)} \rho \circ \gamma^{-1}(z)\,dx\,dy \le \frac{1}{|\gamma(V)|}\iint_{\gamma(V)} \rho \circ \gamma^{-1}(t)\,dx\,dy. \tag{3.11}$$

By (3.8), (3.10), and (3.11), we obtain

$$\frac{|\gamma(U_1 \backslash J_l)|}{|\gamma(U_1)|} \le \frac{1}{|\gamma(V)|}\iint_{\gamma(V)} \rho \circ \gamma^{-1}(z)\,dx\,dy.$$

We will prove

(3.12)

$$\frac{1}{|\gamma(V)|}\int_{\gamma(V)} \rho \circ \gamma^{-1}(z)\,dx\,dy \le \frac{9}{(1-\sqrt{1-\delta_2/2})^2 l^2} + \frac{C_3}{\pi(1-\delta_2/2)l^2}$$

and hence (3.7) holds. For this, consider first the special case $\gamma = \gamma_1 : V \to D$, $\gamma_1(z) = 3/z$. Then,

$$\begin{aligned}
&\frac{1}{|\gamma_1(V)|}\iint_{\gamma_1(V)} \rho\circ\gamma_1^{-1}\,dx\,dy \\
&\quad= \frac{1}{\pi}\iint_D \min\left(\frac{C_3}{\pi[(l-3/|z|)^+]^2},\,1\right)dx\,dy \\
&\quad\le \frac{1}{\pi}\iint_{\{|z|<3/(1-\sqrt{1-\delta_2/2})l\}} 1\cdot dx\,dy \\
&\qquad+ \frac{1}{\pi}\iint_{\{1/(1-\sqrt{1-\delta_2/2})l\le|z|<1\}} \frac{C_3\,dx\,dy}{\pi[(l-3/|z|)^+]^2} \\
&\quad\le \frac{9}{(1-\sqrt{1-\delta_2/2})^2 l^2} + \frac{1}{\pi^2}\iint_D \frac{C_3}{(1-\delta_2/2)l^2}\,dx\,dy.
\end{aligned}$$

Thus (3.12) holds for $\gamma = \gamma_1$. For an arbitrary $\gamma$, as $\gamma(U_1) \supseteq \gamma(V)$ is bounded, we may assume (after composing $\gamma$ with an affine mapping) that $\gamma(V) = D$. Let $h = \gamma_1\circ\gamma^{-1}$, and let $\rho_1 = \rho\circ\gamma_1^{-1}$. Then $h$ is a Möbius transformation of $D$, and $\rho\circ\gamma^{-1} = (\rho\circ\gamma_1^{-1})\circ h = \rho_1\circ h$. The function $\rho_1 : D \to [0, 1]$ depends only on $|z|$ and is nonincreasing in $|z|$. Thus (3.12) (and hence (3.7)) follows by the following lemma.

**Lemma 3.4.** *Let $\rho_1(z) = \rho_1(|z|) : D \to [0, 1]$ be a function which depends only on $|z|$ and is nonincreasing in $|z|$. For any Möbius transformation $h$ of $D$, we have*

$$\iint_D \rho_1\circ h(z)\,dx\,dy \le \iint_D \rho_1(z)\,dx\,dy. \tag{3.13}$$

*Proof.* Since $h^{-1}(\{|z|\le r\})$ has (Euclidean) radius $\le r$ for any $r\in(0, 1]$, we deduce that (3.13) holds for the characteristic functions of disks centered at $0$, and hence for their linear combinations with positive coefficients. But any function $\rho_1$ described in the lemma can be approximated in $L^1$-norm by these linear combinations, so (3.13) holds for $\rho_1$.

*Proof of Lemma* 3.1. Let

$$C_3 = \max\left\{4\pi,\ \frac{18\pi(1-\delta_2)(1-\delta_2/2)}{(1-\sqrt{1-\delta_2/2})^2\delta_2}\right\}. \tag{3.14}$$

Then (3.1) is trivial for $m\le 2$. Let $l\ge 3$, and assume that (3.1) holds for $m<l$. We prove it holds for $m=l$. By Lemma 3.3 and (3.14), we obtain (3.6), which implies (3.1) in virtue of (3.4) and (3.5).

**Remark.** Let $G_\infty$ be the Schottky group generated by inversions on the circles of the infinite regular hexagonal circle packing $H_\infty = \bigcup_{n=1}^{\infty} H_n$.

Then it is clear that $G_\infty$ is a Kleinian group with a fundamental domain $I_\infty$ formed by all interstices bounded by the circles of $H_\infty$. Clearly $J_n \subseteq \bigcup_{h\in G_\infty} g(I_\infty)$ and by Lemma 3.1 we deduce that $|\mathbf{C}\setminus \bigcup_{g\in G_\infty} g(I_\infty)| = 0$, so the limit set of $G_\infty$ has measure zero. More generally, let $P$ be any circle packing on the sphere $\widehat{\mathbf{C}}$ which satisfies the following conditions:

(i) there is no circle of $P$ lying in the interstice bounded by any three mutually tangent circles of $P$ and,

(ii) the circles of $P$ which are tangent to any given circle of $P$ form a closed chain and their number is bounded by some uniform constant.

Let $G$ be the Schottky group generated by the inversions on the circles of $P$. Then the limit set of $G$ has measure zero. To prove this fact, let $I$ be the complement (in $\widehat{\mathbf{C}}$) of the union of the disjoint disks bounded by circles of $P$. Clearly $I$ is a nonvoid fundamental domain for $G$. By an argument similar to the proof of Lemma 3.2, we may deduce that for any Möbius transformation $h$, the measure of the image by $h$ of $\bigcup_{g\in G} g(I)$ in any disk bounded by a circle of $P$ has a positive ratio uniformly bounded away from zero. But any limit point of $G$ lies in infinitely many images of these disks by the elements of $G$, and therefore any limit point is not a Lebesgue point for the limit set. So the limit set has measure zero. It is worth pointing out that condition (i) is essential, as is shown by the fact that the limit set of the Apollonian packing has full measure on $S^2$.

## 4. Globally uniform convergence of quasiconformal mappings

Consider a sequence of $C$-quasiconformal mappings $f_n : \Omega_n \to \Omega'_n$, where $C \geq 1$, and $\Omega_n$ and $\Omega'_n$ are some sequences of (open) Jordan domains in $\widehat{\mathbf{C}}$ converging in the sense of Carathéodory to some Jordan domains $\Omega$ and $\Omega'$ respectively. Suppose that $f_n$ converges uniformly on compact subsets of $\Omega$ to some quasiconformal mapping $f : \Omega \to \Omega'$, and the complex dilations $\lambda_n$ of $f_n$ converge almost everywhere pointwise to the complex dilation $\lambda$ of $f$. One would ask under what conditions does $f_n$ converge globally uniformly to $f$? Here, the globally uniform convergence of $f_n$ means that for any $\varepsilon > 0$, there is some $n(\varepsilon)$ and $\delta(\varepsilon) > 0$ such that for any $n \geq n(\varepsilon)$, $z \in \Omega_n$, and $w \in \Omega$ with $|z - w| \leq \delta(\varepsilon)$, we have

$$|f_n(z) - f(w)| \leq \varepsilon. \tag{4.1}$$

If $f_n : \Omega_n \to \Omega'_n$ and $g_n : \Omega'_n \to \Omega''_n$ converge globally uniformly to $f : \Omega \to \Omega'$ and $g : \Omega' \to \Omega''$ respectively, then $g_n \circ f_n$ converges globally uniformly to $g \circ f$. If $f_n: \Omega_n \to \Omega'_n$ converges to $f: \Omega \to \Omega'$ which extends to a homeomorphism between $\overline{\Omega}$ and $\overline{\Omega}'$, then $f_n^{-1}: \Omega'_n \to \Omega_n$ converges to $f^{-1}: \Omega' \to \Omega$.

We will give a sufficient condition for the globally uniform convergence of $f_n$. As a consequence, we will show that in the case of Jordan domains the mappings $f_\varepsilon$ constructed in [7] converge globally uniformly to the Riemann mapping, and the approximating solutions $f_\delta$ of [4] also converge globally uniformly to the solution of the Beltrami equation.

For any two Jordan domains $\Omega_0$ and $\Omega_1$ in $\widehat{\mathbf{C}}$, we define the distance $\rho(\Omega_0, \Omega_1)$ by

(4.2)

$$\rho(\Omega_0, \Omega_1) = \inf\left\{ \sup_{z \in \partial\Omega_0} d(\psi(z), z);\ \psi: \partial\Omega_0 \to \partial\Omega_1 \text{ is an orientation preserving homeomorphism} \right\},$$

where $d(\cdot, \cdot)$ denotes the spherical distance between two points of $\widehat{\mathbf{C}} = S^2$. It is easy to check that $\rho$ defines a metric on the set of all Jordan domains of $\mathbf{C}$. Remark that $\rho(\Omega_n, \Omega) \to 0$ (so-called "Frechet convergence") is stronger than Carathéodory convergence. The following lemma follows immediately by [8, Theorem V].

**Lemma 4.1.** *Let $\Omega_n$ and $\Omega$ be Jordan domains such that $\rho(\Omega_n, \Omega) \to 0$, and let $f_n: D \to \Omega_n$ and $f: D \to \Omega$ be some conformal mappings such that $f_n$ converges locally uniformly to $f$. Then $f_n$ converges globally uniformly to $f$.*

**Corollary 4.2.** *Let $\Omega_n$, $\Omega'_n$, $\Omega$, and $\Omega'$ be some Jordan domains such that $\rho(\Omega_n, \Omega) \to 0$ and $\rho(\Omega'_n, \Omega') \to 0$. Then any sequence of conformal mappings $f_n: \Omega_n \to \Omega'_n$ which converges locally uniformly to some conformal mapping $f: \Omega \to \Omega'$ converges globally uniformly to $f$.*

*Proof.* Let $g_n: D \to \Omega_n$ and $g: D \to \Omega$ be some conformal mappings such that $g(0) \neq \infty$, $\partial_z g(0) > 0$, $g_n(0) \to g(0)$, and $\partial_z g_n(0) > 0$. Then $g_n$ converges locally to $g$. Since $\rho(\Omega_n, \Omega) \to 0$, it follows that Lemma 4.1 that $g_n$ converges globally uniformly to $g$. As $g$ is a conformal homeomorphism between Jordan domains, $g$ extends to a homeomorphism of $\overline{D}$ and $\overline{\Omega}$, so $g_n^{-1}$ converges uniformly to $g^{-1}$.

On the other hand, the mappings $f_n \circ g_n: D \to \Omega'_n$ converge locally to $f \circ g$, and $\rho(\Omega'_n, \Omega') \to 0$. Again by Lemma 4.1 we deduce that $f_n \circ g_n$ converges globally uniformly to $f \circ g$. So $f_n = (f_n \circ g_n) \circ g_n^{-1}$ converges globally uniformly to $f$. q.e.d.

The main result of this section is the following:

**Theorem 4.3.** *Let $\Omega_n$, $\Omega'_n$, $\Omega$, and $\Omega'$ be Jordan domains such that $\rho(\Omega_n, \Omega) \to 0$ and $\rho(\Omega'_n, \Omega') \to 0$. Let $f_n: \Omega_n \to \Omega'_n$ be a sequence of $C$-quasiconformal homeomorphisms which converges locally to a quasiconformal homeomorphism $f: \Omega \to \Omega'$. If the complex dilations $\lambda_n$ of $f_n$ converge almost everywhere pointwise to the complex dilation $\lambda$ of $f$, then $f_n$ converges globally uniformly to $f$.*

*Proof.* We may assume without loss of generality that all domains $\Omega_n$, $\Omega'_n$, $\Omega$, and $\Omega'$ are contained in some disk, say the unit disk $D$. Define $\mu_n$, $\mu: \mathbf{C} \to \{|\zeta| < 1\}$ by

$$\mu_n(z) = \begin{cases} \lambda_n(z) & \text{if } z \in \Omega_n, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\mu(z) = \begin{cases} \lambda(z) & \text{if } z \in \Omega, \\ 0 & \text{otherwise}, \end{cases}$$

respectively. Then since $\lambda_n(z) \to \lambda(z)$ almost everywhere $z \in \Omega$ and $\|\lambda_n\| < 1$, we deduce that for any $p > 2$,

$$\|\mu_n - \mu\|_p = \left(\int\int_{\mathbf{C}} |\mu_n(z) - \mu(z)|^p \, dx\, dy\right)^{1/p} \to 0. \tag{4.3}$$

Define for each $p \geq 2$ the operators $P: L^p(\mathbf{C}; \mathbf{C}) \to C^{1-2/p}(\mathbf{C}; \mathbf{C})$ and $T: L^p(\mathbf{C}; \mathbf{C}) \to L^p(\mathbf{C}; \mathbf{C})$ by

$$(Ph)(\zeta) = -\frac{1}{\pi} \iint_C h(z) \left(\frac{1}{z-\zeta} - \frac{1}{z}\right) dx\, dy$$

and

$$(TH)(\zeta) = \partial_{\bar{\zeta}}(Ph)(\zeta) = \lim_{\varepsilon \to 0} -\frac{1}{\pi} \iint_{|z0\zeta|>\varepsilon} \frac{h(z)}{(z-\zeta)^2} \, dx\, dy.$$

It is well known that $\|T\|_2 = 1$ and $\|T\|_p \to 1$ for $p \to 2$ (see [1], [3]).

Fix some $p > 2$ small enough so that $\|T\|_p \|\mu^h\|_\infty$ are uniformly bounded by a constant smaller than $1$. For any complex dilation $\mu \in L^\infty(\mathbf{C}; C)$, denote by $h^\mu \in L^p(\mathbf{C}; \mathbf{C})$ the solution to

$$h^\mu = T(\mu h^\mu) + T\mu. \tag{4.4}$$

By [1, p. 90–92], $h$ exists and is unique, and the mapping $f^\mu(z) = z + P(\mu(h^\mu + 1))(z)$ is a quasiconformal homeomorphism of $\mathbf{C}$ with complex dilation $\mu$. We have

$$\partial_{\bar{z}} f^\mu = \mu(h^\mu + 1) \quad \text{and} \quad \partial_z f^\mu = h^\mu + 1.$$

Now let $h_n = h^{\mu_n}$, $h = h^{\mu}$, $g_n = f^{\mu_n}$, and $g = f^{\mu}$. Then

$$\partial_z(g_n(z) - g(t)) = h_n - h$$

and

$$\partial_{\bar{z}}(g_n(z) - g(z)) = \mu_n h_n - \mu h.$$

But as $\mu_n \to \mu$ in $L^p(\mathbf{C}, \mathbf{C})$ we have $h_n \to h$ in $L^p(\mathbf{C}, \mathbf{C})$. Then $\partial_z(g_n(z) - g(z)) \to 0$ in $L^p(\mathbf{C}, \mathbf{C})$. On the other hand, $g_n(0) - g(0) = 0$. Hence $g_n$ converges to $g$ in $C^{1-2/p}(\overline{D})$ and $g \in C^{1-2/p}(\overline{D})$. This implies that $g_n|\overline{D}$ converges globally uniformly to $g|\overline{D}$. By the assumption $\rho(\Omega_n, \Omega) \to 0$, it follows that $\rho(g_n(\Omega_n), g(\Omega)) \to 0$. Let $h_n = f_n \circ g_n^{-1}: g_n(\Omega_n) \to \Omega_n'$ and let $h = f \circ g^{-1}: g(\Omega) \to \Omega'$. Clearly $h_n$ is a sequence of conformal mappings which converges locally to $h$. Using Corollary 4.2, we deduce that $h_n$ converges globally uniformly to $h$. So $f_n = h \circ g_n$ converges globally uniformly to $f$.

**Corollary 4.4.** (i) *Suppose $R$ is a Jordan domain. The circle packing solutions $f_\varepsilon$ of* [7] *converge globally uniformly to the Riemann mapping.*

(ii) *The approximating solutions $f_\delta: \Omega \to \mathbf{C}$ of* [4] *converge globally uniformly to the solution of the Beltrami equation.*

*Proof.* (i) It is shown in [7] that $f_\varepsilon: R_\varepsilon = |T_\varepsilon| \to D_\varepsilon = |T_\varepsilon'|$ converges uniformly on compact subsets to the Riemann mapping $f: R \to D$, and the complex dilations of $f_\varepsilon$ converge almost everywhere pointwise to $0$. On the other hand, since $R$ is a Jordan domain, $R$ can be approximated (in the $\rho$-metric) by polyhedral domains. From the construction of $R_\varepsilon$ it is then easy to show that $\rho(R_\varepsilon, R) \to 0$. Similarly, by the Length-Area Lemma of [7], we have $\rho(D_\varepsilon, D) \to 0$. We may apply Theorem 4.3 to conclude that $f_\varepsilon$ converges globally uniformly to $f$.

(ii) As $\Omega$ is a Jordan domain in $\mathbf{C}$, from the construction of $\overline{\Omega}_\delta$ it is elementary to show that $\rho(\Omega_\delta, \Omega) \to 0$. As in the proof of (i), we have $\rho(D_\delta, D) \to 0$, where $D_\delta = f_\delta'(\Omega_\delta)$ and $f_\delta = f_\delta'|_\Omega$. As $f_\delta': \Omega_\delta \to D_\delta$ converges locally uniformly to the solution $f$ of the Beltrami equation and their complex dilations converge to $\lambda$ (see [4]), we conclude from Theorem 4.3 that $f_\delta'$ (and hence $f_\delta$) converge globally uniformly to $f$. q.e.d.

The conditions of Theorem 4.3 are quite general. In fact, it can be shown directly that: If $f_n: \Omega_n \to \Omega_n'$ converges globally uniformly to $f: \Omega \to \Omega'$, then $\rho(\Omega_n, \Omega) \to 0$ implies $\rho(\Omega_n', \Omega) \to 0$. In particular, if $\Omega_n = \Omega$ for all $n$, then the globally uniform convergence of $f_n$ implies $\rho(\Omega_n', \Omega) \to 0$.

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